

# $q$ -Moments remove the degeneracy associated with the inversion of the $q$ -Fourier transform

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## Abstract

It was recently proven [Hilhorst, JSTAT, P10023 (2010)] that the  $q$ -generalization of the Fourier transform is not invertible in the full space of probability density functions for  $q > 1$ . It has also been recently shown that this complication disappears if we dispose of the  $q$ -Fourier transform not only of the function itself, but also of all of its shifts [Jauregui and Tsallis, Phys. Lett. A **375**, 2085 (2011)]. Here we show that another road exists for completely removing the degeneracy associated with the inversion of the  $q$ -Fourier transform of a given probability density function. Indeed, it is possible to determine this density if we dispose of some extra information related to its  $q$ -moments.

## 1 Introduction

Nonextensive statistical mechanics [1], a current generalization of the Boltzmann-Gibbs theory, is actively studied in diverse areas of physics and other sciences [2, 3]. This theory is based on a nonadditive entropy, commonly denoted by  $S_q$ , that depends, in addition to the probabilities of the microstates, on a real parameter  $q$ , which is inherent to the system and makes  $S_q$  extensive. In the limit  $q \rightarrow 1$ , nonextensive statistical mechanics yields the Boltzmann-Gibbs theory. This new theory has successfully described many physical and computational experiments. Such systems typically are nonergodic ones, with long-range interactions, long memory and/or other nontrivial ingredients: see for example [4, 5, 6, 7, 8, 9, 10, 11, 12].

The development of nonextensive statistical mechanics introduced, in addition to the generalization of some physical concepts like the Boltzmann-Gibbs-Shannon-von Neumann entropy, the generalization of some mathematical concepts. Remarkable ones are the generalizations of the classical central limit theorem and the Lévy-Gnedenko one. These extensions

are based on a generalization of the Fourier transform (FT), namely the  $q$ -Fourier transform ( $q$ -FT) [13, 14]. These generalized theorems respectively establish, for  $q > 1$ ,  $q$ -Gaussians and  $(q, \alpha)$ -stable distributions as attractors when the considered random variables are correlated in a special manner.

If  $1 < q < 3$ , a  $q$ -Gaussian is a generalization of a Gaussian defined as a function  $G_{q,\beta} : \mathcal{R} \rightarrow \mathcal{R}$  such that

$$G_{q,\beta}(x) = \frac{\sqrt{\beta}}{C_q[1 + (q-1)\beta x^2]^{\frac{1}{q-1}}} \equiv \frac{\sqrt{\beta}}{C_q} \exp_q(-\beta x^2), \quad (1)$$

where  $\beta > 0$  and  $C_q$  is a normalization constant given by

$$C_q = \frac{\sqrt{\pi}\Gamma(\frac{3-q}{2(q-1)})}{\sqrt{q-1}\Gamma(\frac{1}{q-1})}. \quad (2)$$

A  $q$ -Gaussian is not normalizable for  $q \geq 3$ . Its variance is finite for  $q < 5/3$ ; above this value, it diverges. When correlations can be neglected,  $q \rightarrow 1$ , and  $G_{q,\beta}(x) \rightarrow (\beta/\pi)^{1/2} \exp(-\beta x^2)$ , which is a Gaussian.

The  $q$ -FT of a non-negative measurable function  $f$  with support  $\text{supp } f \subset \mathcal{R}$ , denoted by  $F_q[f]$ , is defined, for  $1 \leq q < 3$ , as

$$F_q[f](\xi) = \int_{\text{supp } f} f(x) \exp_q(i\xi x[f(x)]^{q-1}) dx, \quad (3)$$

where  $\exp_q(ix) = \text{pv} [1 + (1-q)ix]^{1/(1-q)}$  for any real number  $x$ , being *pv* the notation for *principal value*. This is a non-linear integral transform when  $q > 1$ . Its relevance in [13] is that it transforms a  $q$ -Gaussian into another one. Hence the  $q$ -FT is invertible in the space of  $q$ -Gaussians [15]. However, it was recently proven, by means of counterexamples, that the  $q$ -FT is not invertible in the full space of probability density functions (pdf's) [16]. In connection to this problem, it is worthy mentioning that it has been found an interesting property of the  $q$ -FT which enables the determination of a given pdf from the knowledge of the  $q$ -FT of an arbitrary translation of such pdf [17].

Here we will discuss the counterexamples given in [16], and we will show that it is possible to determine the pdf's considered in the counterexamples from the knowledge of their  $q$ -FT and some extra information related with their  $q$ -moments, defined here below.

Let  $Q$  be a real number and  $f$  be a pdf of some random variable  $X$  such that the quantity

$$\nu_Q[f] = \int_{\text{supp } f} [f(x)]^Q dx \quad (4)$$

is finite. Then, we can define an *escort* pdf [18] for  $X$ , denoted by  $f_Q$ , as follows:

$$f_Q(x) = \frac{[f(x)]^Q}{\nu_Q[f]}. \quad (5)$$

The moments of  $f_Q$ , which are called  $Q$ -moments of  $f$ , are given by

$$\Pi_Q^{(n)}[f] = \int_{\text{supp } f} x^n f_Q(x) dx = \frac{\mu_Q^{(n)}[f]}{\nu_Q[f]}, \quad (6)$$

where  $\mu_Q^{(n)}[f]$  is the *unnormalized*  $n$ th  $Q$ -moment of  $f$ , defined as follows:

$$\mu_Q^{(n)}[f] = \int_{\text{supp } f} x^n [f(x)]^Q dx, \quad (7)$$

$n$  being a positive integer.

The characteristic function of  $X$  is basically given by the Fourier transform of  $f$ ,  $F[f]$ . It is well known that all the moments of  $f$  can be obtained from the successive derivatives of the characteristic function of  $X$  at the origin. It was shown that the successive derivatives of the  $q$ -FT of  $f$  at the origin are related to specific unnormalized  $Q$ -moments of  $f$  by the following equation [19]:

$$\left. \frac{d^n F_q[f](\xi)}{d\xi^n} \right|_{\xi=0} = i^n \left\{ \prod_{j=1}^{n-1} [1 + j(q-1)] \right\} \mu_{q_n}^{(n)}[f], \quad (8)$$

where  $q_n = nq - (n-1)$ . We can see from this relation that, if the  $q$ -FT of  $f$  does not depend on a certain parameter that appears in  $f$ , then the unnormalized  $n$ th  $q_n$ -moments also do not depend on such parameter. Therefore, these unnormalized moments are unable to identify the pdf  $f$  from its  $q$ -FT. As it will become soon clear, this difficulty does *not* exist for the set of  $\{\nu_q\}$ , which will then provide the desired identification procedure.

## 2 Hilhorst's examples

We discuss in this section two examples proposed by Hilhorst [16], where the pdf depends on a certain real parameter, which disappears when we take its  $q$ -FT. Therefore, at the step of looking at the inverse  $q$ -FT, we face an infinite degeneracy. Next we illustrate, in both examples, how the degeneracy is removed through the values of the  $\{\nu_q\}$ .

### 2.1 First example

Let us consider the function  $h_{q,\lambda,a} : \mathcal{R} \rightarrow \mathcal{R}$  such that [16]

$$h_{q,\lambda,a}(x) = \left( \frac{\lambda}{|x|} \right)^{\frac{1}{q-1}} \quad (9)$$

if  $a < |x| < b$ , where  $q > 1$ , and  $(a, b, \lambda)$  are positive real numbers; otherwise  $h_{q,\lambda,a}(x) = 0$  (see Fig. 1). We can impose the following normalization condition for this function:

$$\int_{-\infty}^{+\infty} h_{q,\lambda,a}(x) dx = 1. \quad (10)$$

From this, it follows that one parameter among  $q, \lambda, a, b$  depends on the other ones. Choosing  $b$  as the dependent parameter, we get

$$b = \begin{cases} \left[ \frac{q-2}{2(q-1)} \lambda^{\frac{1}{1-q}} + a^{\frac{q-2}{q-1}} \right]^{\frac{q-1}{q-2}} & \text{if } q \neq 2 \\ a e^{\frac{1}{2\lambda}} & \text{if } q = 2. \end{cases} \quad (11)$$

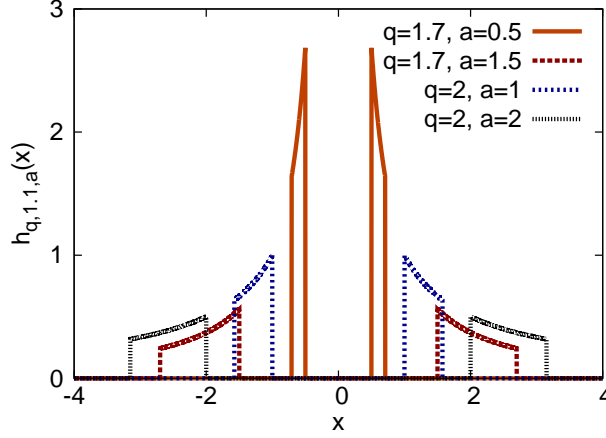


Figure 1: Representation of  $h_{q,\lambda,a}$  for  $\lambda = 1.1$  and different values of  $q$  and  $a$ .

Given  $Q$  such that  $1 \leq Q < 3$ , the  $Q$ -FT of  $h_{q,\lambda,a}$  can be easily reduced to the following expression:

$$F_Q[h_{q,\lambda,a}](\xi) = 2 \int_a^b \left( \frac{\lambda}{x} \right)^{\frac{1}{q-1}} \cos_Q \left( \xi x \left( \frac{\lambda}{x} \right)^{\frac{Q-1}{q-1}} \right) dx, \quad (12)$$

where  $\cos_q$  is the  $q$ -generalization of the trigonometric function  $\cos$  which is defined by [20]

$$\cos_q(x) = \Re(\exp_q(ix)) = \frac{\cos(\frac{1}{q-1} \arctan((q-1)x))}{[1 + (q-1)^2 x^2]^{\frac{1}{2(q-1)}}}. \quad (13)$$

It is easy to notice from (12) that the  $Q$ -FT of  $h_{q,\lambda,a}$  depends on  $a$  if  $Q \neq q$ . However, it does not depend on  $a$  when  $Q = q$  (see Fig. 2), when it is given by  $F_q[h_{q,\lambda,a}](\xi) = \cos_q(\xi\lambda)$ .

Consequently, there exist infinite functions  $h_{q,\lambda,a}$  with the same  $q$  and  $\lambda$  but different  $a$ , which have the same  $q$ -FT. Therefore, it is not possible to determine  $h_{q,\lambda,a}$  just from the knowledge of its  $q$ -FT. However, it may be possible to obtain  $h_{q,\lambda,a}$  from its  $q$ -FT *and* some extra information. For example, we would be able to determine  $h_{q,\lambda,a}$  if we knew the  $q$ -FT of an arbitrary translation of  $h_{q,\lambda,a}$  [17]. Here we will give another approach to this problem.

As  $h_{q,\lambda,a}$  is a non-negative function, which obeys the normalization condition (10), it can be interpreted as a pdf of some random variable. Moreover, for any real number  $Q$ , we have that

$$\nu_Q[h_{q,\lambda,a}] = \begin{cases} \frac{2(q-1)}{q-1-Q} \lambda^{\frac{Q}{q-1}} (b^{\frac{q-1-Q}{q-1}} - a^{\frac{q-1-Q}{q-1}}) & \text{if } Q \neq q-1 \\ 2\lambda \ln(\frac{b}{a}) & \text{if } Q = q-1 \end{cases} \quad (14)$$

is finite. Being  $n$  an even positive integer, we have also that the unnormalized  $n$ th  $Q$ -moment of  $h_{q,\lambda,a}$  is given by

$$\mu_Q^{(n)}[h_{q,\lambda,a}] = \begin{cases} \frac{2(q-1)}{(n+1)(q-1)-Q} \lambda^{\frac{Q}{q-1}} [b^{\frac{(n+1)(q-1)-Q}{q-1}} - a^{\frac{(n+1)(q-1)-Q}{q-1}}] & \text{if } Q \neq (n+1)(q-1) \\ 2\lambda^{n+1} \ln(\frac{b}{a}) & \text{if } Q = (n+1)(q-1). \end{cases} \quad (15)$$

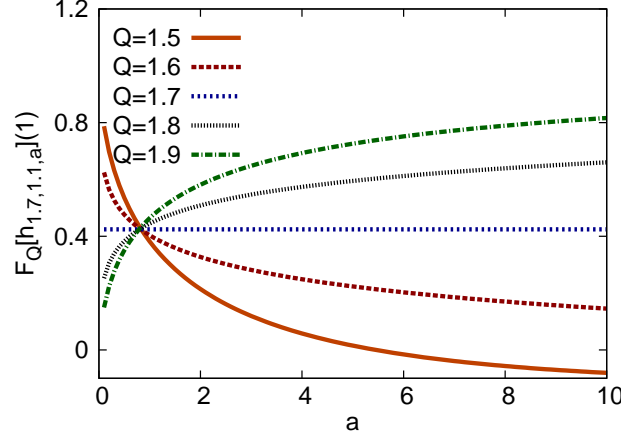


Figure 2: The dependence on  $a$  of  $F_Q[h_{1.7,1.1,a}](1)$  for different values of  $Q$ .

Then, finally, the  $n$ th  $Q$ -moment of  $h_{q,\lambda,a}$  is given by

$$\Pi_Q^{(n)}[h_{q,\lambda,a}] = \begin{cases} (b^n - a^n)[n \ln(\frac{b}{a})]^{-1} & \text{if } Q = q - 1 \\ \frac{na^n b^n}{b^n - a^n} \ln(\frac{b}{a}) & \text{if } Q = (n+1)(q-1) \\ \frac{(q-1-Q)}{(n+1)(q-1)-Q} \left[ \frac{b^{\frac{(n+1)(q-1)-Q}{q-1}} - a^{\frac{(n+1)(q-1)-Q}{q-1}}}{b^{\frac{q-1-Q}{q-1}} - a^{\frac{q-1-Q}{q-1}}} \right] & \text{otherwise.} \end{cases} \quad (16)$$

It is clear that  $\mu_Q^{(m)}[h_{q,\lambda,a}] = 0$  and  $\Pi_Q^{(m)}[h_{q,\lambda,a}] = 0$  for any odd positive integer  $m$ , since  $h_{q,\lambda,a}(x)$  is an even function.

As the  $q$ -FT of  $h_{q,\lambda,a}$  does not depend on  $a$ , then, according to (8), the  $n$ th  $q_n$ -moment of  $h_{q,\lambda,a}$  does not depend on  $a$  either, where  $q_n = nq - (n-1)$ . In fact, if  $q \neq 2$ , we have that

$$\mu_{q_n}^{(n)}[h_{q,\lambda,a}] = \frac{2(q-1)}{q-2} \lambda^{\frac{nq-(n-1)}{q-1}} (b^{\frac{q-2}{q-1}} - a^{\frac{q-2}{q-1}}). \quad (17)$$

Then, using (11), we obtain that  $\mu_{q_n}^{(n)}[h_{q,\lambda,a}] = \lambda^n$ . If  $q = 2$ , we have that  $\mu_{n+1}^{(n)}[h_{q,\lambda,a}] = 2\lambda^{n+1} \ln(b/a)$ , and, using (11), we obtain that  $\mu_{n+1}^{(n)}[h_{q,\lambda,a}] = \lambda^n$ .

While the unnormalized  $Q$ -moments of  $h_{q,\lambda,a}$  may not depend on  $a$  (see Fig. 3), we can straightforwardly verify from (14) that the quantity  $\nu_Q[h_{q,\lambda,a}]$  depends monotonically on  $a$  for any  $Q \neq 1$  (see Fig. 4). The same is true for the normalized  $Q$ -moments (see Fig. 5). Hence, the knowledge of the  $q$ -FT of  $h_{q,\lambda,a}$  and the value of some  $\nu_Q[h_{q,\lambda,a}]$  with  $Q \neq 1$  (extra information) is sufficient to determine the pdf  $h_{q,\lambda,a}$ . We should notice that  $\nu_1[h_{q,\lambda,a}] = 1$  (it does not depend on  $a$ ), then the extra information in this case is trivial.

## 2.2 Second example

Let us consider now the function  $f_{q,A} : \mathcal{R} \rightarrow \mathcal{R}$  such that [16]

$$f_{q,A}(x) = \frac{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{1}{q-2}}}{C_q [1 + (q-1)x^2(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{q-2}}]^{\frac{1}{q-1}}} \quad (18)$$

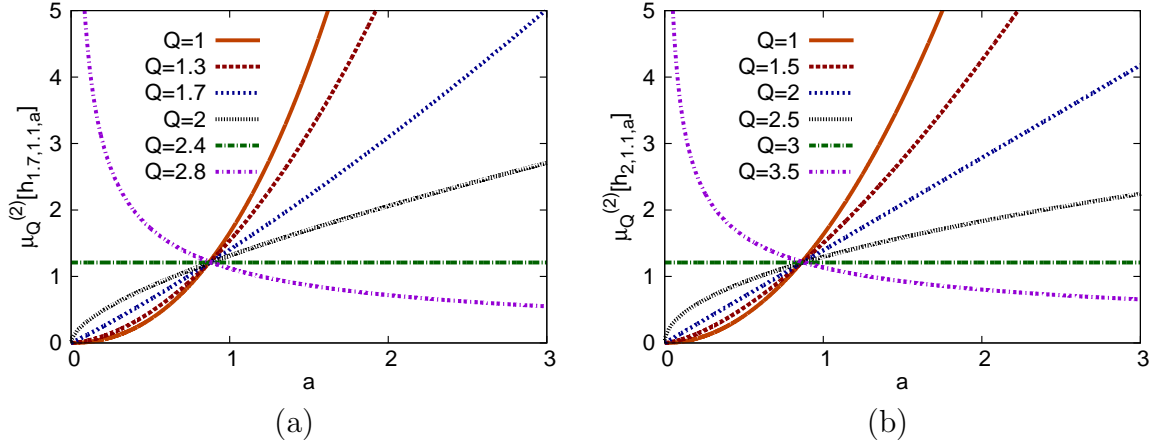


Figure 3: The dependence on  $a$  of the quantities (a)  $\mu_Q^{(2)}[h_{1.7,1.1,a}]$  and (b)  $\mu_Q^{(2)}[h_{2,1.1,a}]$  for different values of  $Q$ .

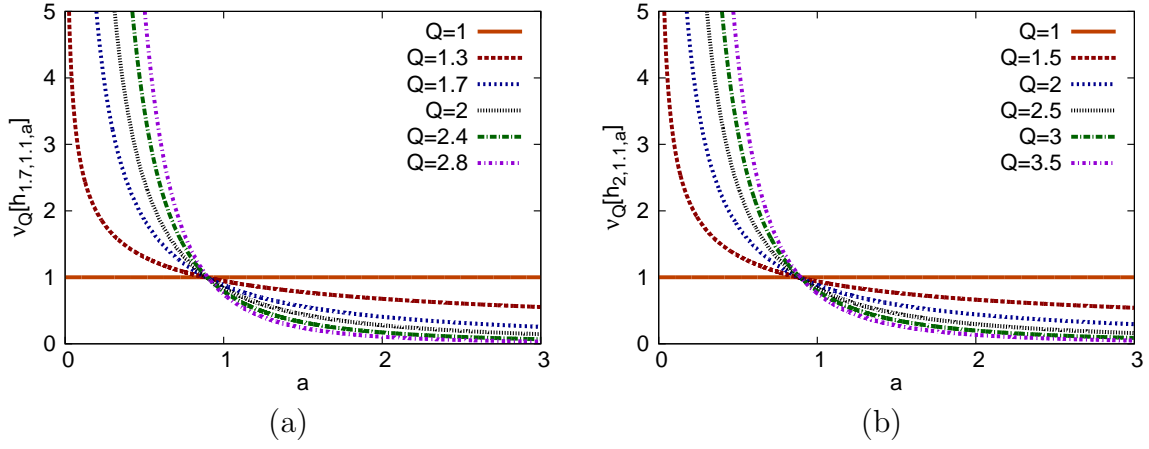


Figure 4: The dependence on  $a$  of the quantities (a)  $\nu_Q[h_{1.7,1.1,a}]$  and (b)  $\nu_Q[h_{2,1.1,a}]$  for different values of  $Q$ .

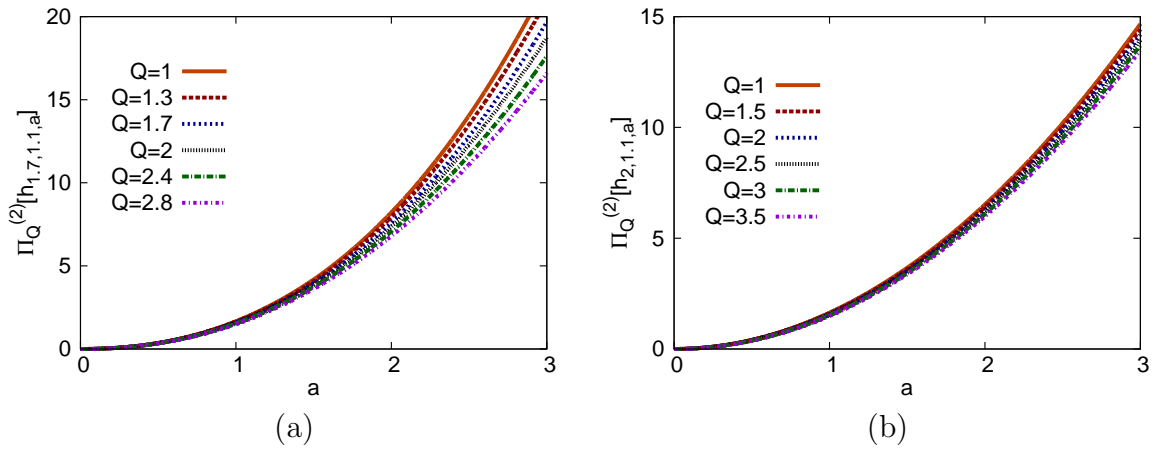


Figure 5: The dependence on  $a$  of the quantities (a)  $\Pi_Q^{(2)}[h_{1.7,1.1,a}]$  and (b)  $\Pi_Q^{(2)}[h_{2,1.1,a}]$  for different values of  $Q$ .

if  $|x|^{(q-2)/(q-1)} > A$ , where  $1 < q < 2$ ,  $A \geq 0$ , and  $C_q$  is the normalization constant of a  $q$ -Gaussian given by (2); otherwise  $f_{q,A}(x) = 0$  (see Fig. 6). We can easily notice that  $f_{q,0}(x) = G_{q,1}(x)$ , where  $G_{q,\beta}(x)$  is defined in (1).

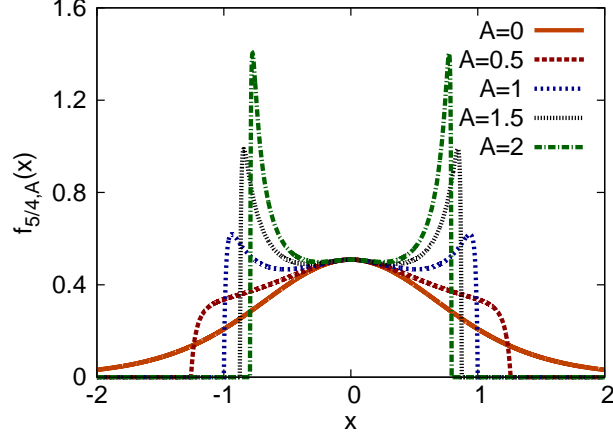


Figure 6: Representation of  $f_{5/4,A}$  for different values of  $A$ .

Let  $1 < Q < 3$  and  $A > 0$ . The  $Q$ -FT of  $f_{q,A}$  is given by (see Fig. 7)

$$F_Q[f_{q,A}](\xi) = \int_{-A^{\frac{q-1}{q-2}}}^{A^{\frac{q-1}{q-2}}} f_{q,A}(x) \exp_Q(i\xi x[f_{q,A}(x)]^{Q-1}) dx. \quad (19)$$

In order to compute this integral in the particular case  $Q = q$ , we should notice first that

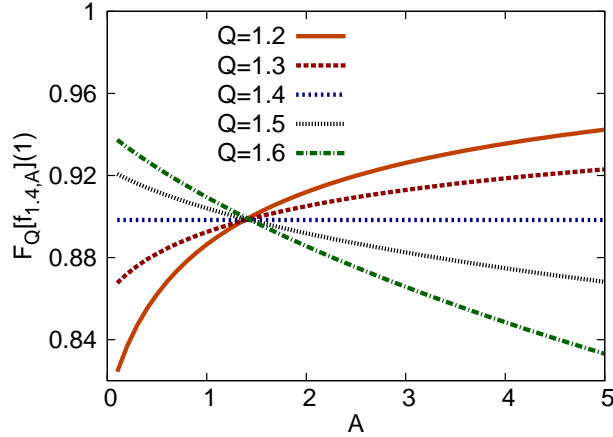


Figure 7: The dependence on  $A$  of  $F_Q[f_{1.4,A}](1)$  for different values of  $Q$ .

$$\begin{aligned} \exp_q(i\xi x[f_{q,A}(x)]^{q-1}) &= \exp_q\left(\frac{i\xi x(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{q-1}{q-2}}}{C_q^{q-1}[1 + (q-1)x^2(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{q-2}}]}\right) \\ &= \frac{[1 + (q-1)x^2(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{q-2}}]^{\frac{1}{q-1}}}{\text{pv} \left\{ 1 - (q-1) \left[ \frac{-x^2}{(1-A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{q-2}}} + \frac{iC_q^{1-q}\xi x}{(1-A|x|^{\frac{2-q}{q-1}})^{\frac{2-q}{q-1}}} \right] \right\}^{\frac{1}{q-1}}} \end{aligned}$$

$$\begin{aligned}
&= [1 + (q-1)x^2(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{q-2}}]^{\frac{1}{q-1}} \\
&\quad \times \exp_q \left( \frac{-x^2}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{2-q}}} + \frac{iC_q^{1-q}\xi x}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{q-1}{2-q}}} \right). \quad (20)
\end{aligned}$$

Then

$$\begin{aligned}
F_q[f_{q,A}](\xi) &= \frac{1}{C_q} \int_{-A^{\frac{q-1}{q-2}}}^{A^{\frac{q-1}{q-2}}} \frac{\exp_q \left( \frac{-x^2}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{2-q}}} + \frac{iC_q^{1-q}\xi x}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{q-1}{2-q}}} \right)}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{1}{2-q}}} dx \\
&= \frac{1}{C_q} \int_{-A^{\frac{q-1}{q-2}}}^{A^{\frac{q-1}{q-2}}} \frac{\exp_q \left( - \left[ \frac{x}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{q-1}{2-q}}} - \frac{iC_q^{1-q}\xi}{2} \right]^2 - \frac{C_q^{2(1-q)}\xi^2}{4} \right)}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{1}{2-q}}} dx. \quad (21)
\end{aligned}$$

Finally, using the change of variables

$$y = \frac{x}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{q-1}{2-q}}} - \frac{iC_q^{1-q}\xi}{2}, \quad (22)$$

we obtain that

$$F_q[f_{q,A}](\xi) = \frac{1}{C_q} \int_{-\infty - \frac{iC_q^{1-q}\xi}{2}}^{+\infty - \frac{iC_q^{1-q}\xi}{2}} \exp_q \left( -y^2 - \frac{C_q^{2(1-q)}\xi^2}{4} \right) dy \quad (23)$$

which does *not* depend on  $A$ . Moreover, the RHS of (23) is equal to the  $q$ -FT of the  $q$ -Gaussian  $G_{q,1}$  (see details in [13]), which, naturally, does not depend on  $A$ . Then, the knowledge of only the  $q$ -FT of  $f_{q,A}$  would not be sufficient information to determine  $f_{q,A}$ . Hence, as in the first example, extra information is needed.

Let  $Q$  be a real number. Considering  $f_{q,A}$  as a pdf of some random variable, we have that

$$\begin{aligned}
\nu_Q[f_{q,A}] &= \int_{-A^{\frac{q-1}{q-2}}}^{A^{\frac{q-1}{q-2}}} \frac{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{Q}{q-2}}}{C_q^Q [1 + (q-1)x^2(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{q-2}}]^{\frac{Q}{q-1}}} dx \\
&= \frac{1}{C_q^Q} \int_{-A^{\frac{q-1}{q-2}}}^{A^{\frac{q-1}{q-2}}} \frac{\left[ \exp_q \left( \frac{-x^2}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{2-q}}} \right) \right]^Q}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{Q}{2-q}}} dx, \quad (24)
\end{aligned}$$

which is finite and depends on  $A$  when  $Q \neq 1$  (see Fig. 8). The unnormalized  $n$ th  $Q$ -moment of  $f_{q,A}$  for any positive integer  $n$  is given by

$$\mu_Q^{(n)}[f_{q,A}] = \frac{1}{C_q^Q} \int_{-A^{\frac{q-1}{q-2}}}^{A^{\frac{q-1}{q-2}}} \frac{x^n \left[ \exp_q \left( \frac{-x^2}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{2(q-1)}{2-q}}} \right) \right]^Q}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{Q}{2-q}}} dx, \quad (25)$$



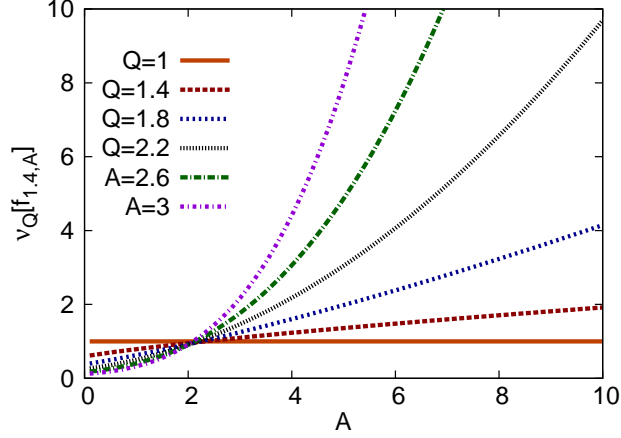


Figure 8: The dependence on  $A$  of the quantity  $\nu_Q[f_{1.4,A}]$  for different values of  $Q$ .

which depends on  $A$  except when  $Q = q_n = nq - (n - 1)$  (see Fig. 9). In this case, using the change of variables

$$y = \frac{x}{(1 - A|x|^{\frac{2-q}{q-1}})^{\frac{q-1}{2-q}}}, \quad (26)$$

we obtain that

$$\mu_{q_n}^{(n)}[f_{q,A}] = \int_{-\infty}^{+\infty} y^n \left[ \frac{1}{C_q} \exp_q(-y^2) \right]^{nq-(n-1)} dy, \quad (27)$$

which is equal to the unnormalized  $n$ th  $q_n$ -moment of the  $q$ -Gaussian  $G_{q,1}$ . Therefore, we see that, like in the first example, the knowledge of any  $\nu_Q[f_{q,A}]$  with  $Q \neq 1$  enables the determination of the pdf  $f_{q,A}$  from its  $q$ -FT.

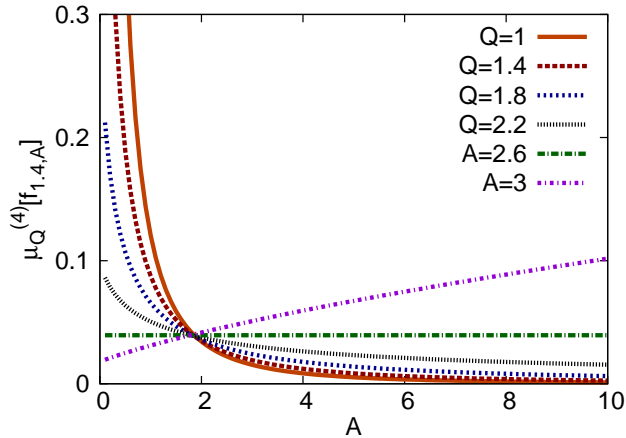


Figure 9: The dependence on  $A$  of the unnormalized fourth  $Q$ -moment of  $f_{1.4,A}$  for different values of  $Q$ .

### 3 Conclusions

Both functions  $h_{q,\lambda,a}$  and  $f_{q,A}$  show that the  $q$ -FT is not invertible in the full space of pdf's, since their  $q$ -FT's do not depend on  $a$  and  $A$  respectively. However, if  $Q \neq q$ , this problem would not occur for the  $Q$ -FT of both functions (see Figs. 2 and 7). In other words, the  $Q$ -FT of both functions with  $Q \neq q$  would in principle be invertible. Furthermore, in the case  $Q = q$ , Figs. 4 and 8 show that the quantities  $\nu_Q[h_{q,\lambda,a}]$  and  $\nu_Q[f_{q,A}]$  depend monotonically on  $a$  and  $A$  respectively, which removes the degeneracy. Therefore, the knowledge of the  $q$ -FT of both functions and a single value of  $\nu_Q[h_{q,\lambda,a}]$  and  $\nu_Q[f_{q,A}]$  is sufficient to determine the functions  $h_{q,\lambda,a}$  and  $f_{q,A}$ .

If we were in the case that a pdf  $f$  depends on two or more parameters and its  $q$ -FT does not depend on more than one of such parameters, we would expect this method of identification of the inverse  $q$ -FT to work as fine as in the case of the functions considered in this paper. However, it might be possible that more than one value of  $\nu_Q$  is needed.

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